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Determination of the apsidal angles and Bertrand's theorem

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We derive an expression for the determination of the apsidal angles that holds good for arbitrary central potentials. This formula can be useful in the calculation of precession rates. Then we discuss under what conditions the apsidal angles remain independent of the mechanical energy and angular momentum in the central force problem and, as a consequence, an alternative and nonperturbative proof of Bertrand's theorem is obtained.

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I. INTRODUCTION

In 1873, Bertrand [1] published a short but important paper in which he proved that there were only two central fields for which all orbits radially bounded are closed, namely, the isotropic harmonic-oscillator field and the gravitational one. Because of this additional accidental degeneracy [2]—in the language of group theory, we associated the unitary group U(3) with the harmonic oscillator and the orthogonal group O(4) with the gravitational potential—the properties of those two fields have been under close scrutiny since Newton's [3] times. Newton addresses to the isotropic harmonic oscillator in proposition X Book I of the *Principia* and to the inverse-square law in proposition XI. Newton showed that both fields give rise to elliptical orbits with the difference that in the first case the force is directed toward the geometrical center of the ellipse, and in the second case the force is directed to one of the foci. Bertrand's proof is concise, elegant, and, contrary to what one may be led to think by a number of perturbative demonstrations that can be found in textbooks and papers on the subject, fully nonperturbative. As examples of perturbative demonstrations, the reader can consult Refs. [2,4–6]. We can also find in the literature demonstrations that resemble the spirit of Bertrand's original work as, for example, Refs. [2,7,8]. All perturbative demonstrations and most of the nonperturbative ones, however, have a restrictive feature; that is, they set a limit on the number of possibilities of the existence of central fields with the property mentioned above to a finite number and finally show explicitly that among the surviving possibilities only two—the Newtonian and the isotropic harmonic oscillator—are really possible.

Let us now outline briefly Bertrand's approach to the problem. In his paper, Bertrand initially proved, by taking into consideration, the equal radii limit that a central force $f(r)$ acting on a pointlike body able of generating radially bounded orbits must necessarily be of the form,

$$f(r) = \kappa r^{(1/p^2-3)},$$

where r is the radial distance to the center of force, κ is a constant, and p is a rational number. Next, making use of this particular form of the law of force and considering also an additional limiting condition, Bertrand finally showed that only for $p=1$ and $p=1/2$, which correspond to Newton's gravitational law of force

$$f(r) = -\frac{\kappa}{r^2},$$

and to the isotropic harmonic-oscillator law of force

$$f(r) = -\kappa r,$$

respectively, we can have orbits with the properties stated in the theorem. Moreover, Bertrand can also prove that only for these laws of force, all bounded orbits are closed. For more details, the reader is referred to the original paper or to its English translation [1].

For radially bounded orbits, there are two extreme values for the possible radii, namely, a maximum and a minimum one. These extreme values were named by Newton himself superior apse and inferior apse, respectively, or briefly, apses. The angular displacement between these two successive points or apses defines the apsidal angle $\Delta\theta_a$. In a central field of force, the apsidal angle depends on the total mechanical energy E and the magnitude of the angular momentum ℓ.

In the present paper, our major purpose is to derive a general expression for the apsidal angle in terms of these initial conditions. Putting aside the restriction of closed orbits, we ask ourselves the following question. For what potentials the apsidal angle has the same value for all orbits—that is for arbitrary energy and angular momentum—and

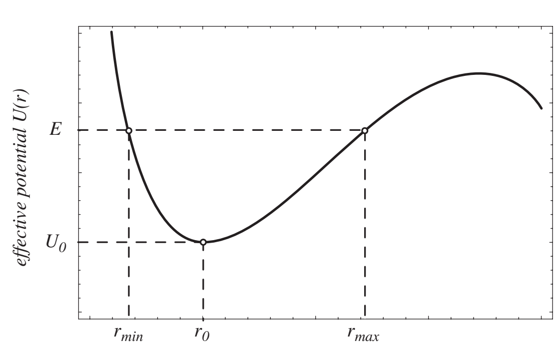

FIG. 1. General form of the effective potential energy.

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what is its value? The answer will show that the apsidal angle is constant for only two potentials: the Newtonian potential and the isotropic harmonic-oscillator one, and consequently, as a corollary, we have an alternative and nonperturbative proof of Bertrand's theorem .

II. APSIDAL ANGLE

In a central field, in which the magnitude of the force $\mathbf{f}(\mathbf{r})$ depends only on the distance $r=\|\mathbf{r}\|$ to the center of force, we can introduce the potential function $V(r)$ with the property $\boldsymbol{f}(\mathbf{r})=-\boldsymbol{\nabla}V(r)$ such that the total mechanical energy E of a particle with mass m orbiting in this field is a constant of motion. Thus, we write

$$E=\frac{m}{2}\boldsymbol{v}^2+V(r), \tag{1}$$

where $\boldsymbol{v}$ is the velocity of the particle. Moreover, the conservation of the angular momentum $\boldsymbol{l}$ of the particle in a central field constrains its motion to a fixed plane and allows for the introduction of an effective potential defined by [9]

$$U(r)=V(r)+\frac{\ell^2}{2mr^2}, \tag{2}$$

with the help of which it is possible to reduce the motion to an equivalent unidimensional problem. For $\ell=0$, the motion is purely radial, but in this case we can evaluate the corresponding apsidal angle by taking the limit $\ell\to 0$ in our general results. Therefore, in what follows, we will consider $\ell \neq 0$.

Let us define for convenience the variable $z=1/r$ and the functions $v(z)=V(1/z)$ and $u(z)=U(1/z)$ in such a way that the effective potential (2) is now written as

$$u(z)=v(z)+\frac{\ell^2}{2m}z^2. \tag{3}$$

At this point it would be natural, in order to have bounded orbits, to suppose that the effective potential has a point of minimum at $z=z_0$. We will assume, however, a less restrictive condition. We will only impose that $u'(z_0)=0$. As we will see below, we will be able to show that z_0 must be an extremum in order to have a force field that could generate well-defined apsidal angles. Therefore, we write

$$u'(z_0)=v'(z_0)+\frac{\ell^2}{m}z_0=0, \tag{4}$$

from which we obtain

$$\ell^2=-mv'(z_0)/z_0. \tag{5}$$

Evidently, Eq. (5) makes sense only for an attractive force field. Taking this result into the effective potential (3), we can write

$$u(z)=v(z)-\frac{v'(z_0)}{2z_0}z^2. \tag{6}$$

In this way, we see that the first derivative of the effective potential at the point $z=z_0$ is zero for each and every potential function $v(z)$. This means that the vanishing of this derivative does not impose restrictions of any kind on the potential function $v(z)$. The second derivative at the point $z=z_0$ is

$$u''(z_0)=v''(z_0)-\frac{v'(z_0)}{z_0}. \tag{7}$$

If $u''(z_0)$ is zero for an arbitrary point z_0, we see that the potential must be of the form $v(z)=a+bz^2$, and in this case the force must be of the form κ/r^3. Newton in the proposition IX of Book I of the Principia showed that in this case, the orbit is an equiangular spiral and therefore without apsidal points. Hence, in what follows, we will suppose that the second derivative of the effective potential is not null at z_0. Then we assume that this extremum of the effective potential is in fact a minimum in order to permit bounded orbits (Fig. 1).

If now we assume that u and z are complex variables and perform an analytical continuation of the function $u(z)$, we can determine the inverse function $z=z(u)$, by applying the generalization of the Bürmann-Lagrange series for a multivalued inverse function [10]. With this purpose in mind, we choose the point $u_0=u(z_0)$ as the point around which we will perform the expansion of this function. The result is

$$z=z_0+\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{d^{n-1}}{d\zeta^{n-1}}\chi^n(\zeta)\right]_{\zeta=z_0}(u-u_0)^{n/2}, \tag{8}$$

where

$$\chi(\zeta)=\frac{\zeta-z_0}{[u(\zeta)-u(z_0)]^{1/2}}, \tag{9}$$

and $u_0=u(z_0)$ is a first-order algebraic branching point of the function $z(u)$. It follows that we can define two real inverse functions of the function $u=u(z)$; the first one is

$$z_1(u)=z_0+\sum_{n=1}^{\infty}C_n(u-u_0)^{n/2}, \tag{10}$$

which holds for $z>z_0$ (or $r<r_0$). The second one is

$$z_2(u)=z_0+\sum_{n=1}^{\infty}(-1)^nC_n(u-u_0)^{n/2}, \tag{11}$$

which holds for $z<z_0$ (or $r>r_0$), and where we have written for convenience

$$C_n=\frac{1}{n!}\left.\frac{d^{n-1}\chi^n(\zeta)}{d\zeta^{n-1}}\right|_{\zeta=z_0}. \tag{12}$$

From Eq. (9) we notice that the functions $z_1(u)$ and $z_2(u)$ will be well-defined real functions only if z_0 is a point of minimum of the effective potential. It is also convenient to write the function (9) in the form

$$\chi(\zeta) = \frac{1}{\sqrt{\frac{v(\zeta) - v(z_0) - v'(z_0)(\zeta - z_0)}{(\zeta - z_0)^2} - \frac{v'(z_0)}{2z_0}}}, \quad (13)$$

which can be obtained by making use of Eq. (6). For radially bounded orbits, the two extreme values for the radii, namely, $r_{\max}$ and $r_{\min}$, maximum and minimum (or $z_{\min}$ and $z_{\max}$), respectively, are determined by the condition $\dot{r}_a=0$ or $\dot{z}_a=0$. The particle oscillates indefinitely between $r_{\max}$ and $r_{\min}$. In terms of the effective potential, radially closed orbits are characterized by extreme points that satisfy the condition $E=U(r_a)$, or $E=u(z_a)$. For convenience, we take the direction defined by the arbitrary vector $\boldsymbol{r}_0$ as the reference for the measure of angular displacements. Hence, the angular displacement between two successive apses, that is, the apsidal angle $\Delta\theta_a$, can be written in the form

$$\Delta\theta_a = \Delta\theta_1 + \Delta\theta_2, \quad (14)$$

where $\Delta\theta_1$ is the angular displacement from the point $\boldsymbol{r}_0$ to the point $\boldsymbol{r}_{\min}$ and $\Delta\theta_2$ is the angular displacement to the apsidal point $\boldsymbol{r}_{\max}$.

The angular displacement in a central field of force can be easily determined from the conservation laws of the mechanical energy and the angular momentum. Hence, we can write

$$\Delta\theta_1 = -\int_{r_0}^{r_{\min}} \frac{\ell}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(E-U)}}, \quad (15)$$

and

$$\Delta\theta_2 = \int_{r_0}^{r_{\max}} \frac{\ell}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}(E-U)}}. \quad (16)$$

Making use of the inverse functions (10) and (11), we have

$$\Delta\theta_1 = \frac{\ell}{\sqrt{2m}} \int_{U_0}^{E} \frac{dz_1}{dU} \frac{dU}{\sqrt{(E-U)}} = \frac{\ell}{2\sqrt{2m}} \sum_{n=1}^{\infty} nC_n \int_{U_0}^{E} \frac{(U-U_0)^{n/2-1}}{\sqrt{E-U}} dU, \quad (17)$$

and

$$\Delta\theta_2 = \frac{\ell}{\sqrt{2m}} \int_{U_0}^{E} \frac{dz_2}{dU} \frac{dU}{\sqrt{(E-U)}} = \frac{\ell}{2\sqrt{2m}} \sum_{n=1}^{\infty} n(-1)^n C_n \int_{U_0}^{E} \frac{(U-U_0)^{n/2-1}}{\sqrt{E-U}} dU. \quad (18)$$

The integrals in Eqs. (17) and (18) can be evaluated with the help of

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\nu-1}dx = (b-a)^{\lambda+\nu-1}B(\lambda,\nu),$$

where

$$B(\lambda,\nu) = \frac{\Gamma(\lambda)\Gamma(\nu)}{\Gamma(\lambda+\nu)},$$

which holds for $b>a$, Re $\mu>0$, and Re $\nu>0$ (see [11], formula 3.196.3). Therefore, we have

$$\Delta\theta_1 = \frac{\ell\pi}{2\sqrt{2m}} \sum_{k=0}^{\infty} (2k+1)C_{2k+1} \frac{(2k-1)!!}{(2k)!!}(E-U_0)^k + \frac{\ell}{2\sqrt{2m}} \sum_{k=0}^{\infty} 2(k+1)C_{2k+2} \frac{2(2k)!!}{(2k+1)!!}(E-U_0)^{k+1/2}, \quad (19)$$

and

$$\Delta\theta_2 = \frac{\ell\pi}{2\sqrt{2m}} \sum_{k=0}^{\infty} (2k+1)C_{2k+1} \frac{(2k-1)!!}{(2k)!!}(E-U_0)^k - \frac{\ell}{2\sqrt{2m}} \sum_{k=0}^{\infty} 2(k+1)C_{2k+2} \frac{2(2k)!!}{(2k+1)!!}(E-U_0)^{k+1/2}. \quad (20)$$

Notice that the first term in the first summation is the only one that does not depend on the energy. Notice also that the constants C_n do not depend on the energy as well. After adding Eqs. (19) and (20), we obtain for the apsidal angle the expression

$$\Delta\theta_a = \frac{\ell\pi}{\sqrt{2m}} \sum_{k=0}^{\infty} (2k+1)C_{2k+1} \frac{(2k-1)!!}{(2k)!!}(E-U_0)^k. \quad (21)$$

Notice that this series does not depend on the even coefficients. Equations (19)–(21) are our main results. These results allow for the explicit calculation of the apsidal angle for given initial conditions. Equation (21) represents the series development of an analytical function and is convergent in some neighborhood of U_0. For practical purposes, Eq. (21) can be truncated and we can deal with a finite number of terms as required by the precision of the measurements. In Sec. III we will use these equations to answer the question we posed at the beginning of this work; that is, for what potentials the apsidal angle has the same value for all orbits, i.e., for arbitrary values of the energy and angular momentum, and what value it assumes.

III. NEWTONIAN POTENTIAL

Let us begin by looking for potentials that keep the partial angles $\Delta\theta_1$ and $\Delta\theta_2$ separately constant. If this happens to be so, the apsidal angle $\Delta\theta_a$ will also be constant for any value of the energy and the angular momentum. In order to accomplish that, it is necessary that for $n\geq 2$ all coefficients C_n in Eqs. (17) and (18) be equal to zero. After evaluating the coefficient C_2 and setting it equal to zero, we obtain $v'''(z_0)=0$, and taking into account that z_0 is an arbitrary point, it follows that

$$v'''(z) = 0. \quad (22)$$

The general solution of Eq. (22) is

$$v(z) = \alpha z^2 + \beta z + \gamma. \tag{23}$$

The integration constant γ is an additive term to the potential and can be discarded without loss of generality. The second derivative of the effective potential at the stationary point is $u''(z_0) = -\beta/z_0$. It follows that z_0 is effectively a minimum only if the constant β is negative and this corresponds to an attractive field. For a potential function of the form given by Eq. (23), we can evaluate the function $\chi(\zeta)$ given by Eq. (13) to obtain

$$\chi(\zeta) = \frac{1}{\sqrt{-\dfrac{\beta}{2z_0}}}. \tag{24}$$

Hence, the only nonzero coefficient is C_1 because the function $\chi(\zeta)$ for this potential does not depend on ζ and therefore $\Delta\theta_1 = \Delta\theta_2 = \text{const}$. Evaluating the first coefficient for this potential, we obtain

$$\Delta\theta_1 = \Delta\theta_2 = \frac{\pi}{2}\sqrt{1 + \frac{2\alpha z_0}{\beta}}. \tag{25}$$

For a fixed value of the angular momentum, any potential of the form given by Eq. (23) generates bounded orbits with equal and constant angles $\Delta\theta_1$ and $\Delta\theta_2$ and thus the apsidal angle $\Delta\theta_a$ as defined by Eq. (14) will also be constant. Moreover, if only the constant α vanishes, that is, if the potential is of the form $v(z) = \beta z$, it is possible to obtain a potential for which the apsidal angle is also independent of the angular momentum. This potential corresponds to the Newtonian one, the apsidal angle is π, and the orbits are closed. The vector $\mathbf{r}_0$ is perpendicular to the vectors $\mathbf{r}_{\max}$ and $\mathbf{r}_{\min}$, which in their turn are antiparallel vectors with respect to each other, defining in this way only one symmetry axis of the trajectory of the particle. As we can see from Eq. (25), it is possible to calculate exactly the apsidal angle for a field given by Eq. (23) even if $\alpha \neq 0$, and in this way determine the precession rate. This suggests that it is also possible to generalize this procedure in order to perform this calculation for an arbitrary central field.

IV. ISOTROPIC HARMONIC-OSCILLATOR POTENTIAL

Let us now look for potentials that keep the apsidal angle $\Delta\theta_a$ constant. In order to accomplish this, it is necessary that all the odd coefficients C_n starting from the third one be zero. As before, we begin by explicitly calculating the lowest-order coefficient, namely, C_3. Evaluating this coefficient, setting it equal to zero, and considering the arbitrariness of z_0, we obtain the following differential equation:

$$\frac{5}{3}v'''^2(z) - \left[v''(z) - \frac{v'(z)}{z}\right]v^{(iv)}(z) = 0. \tag{26}$$

Writing the function $v(z)$ as

$$v(z) = \frac{z^2}{2}\int \frac{\phi(z)}{z}dz - \frac{1}{2}\int z\varphi(z)dz, \tag{27}$$

we can recast Eq. (26) into the form

$$\frac{5}{3}\frac{1}{z\varphi(z)}\frac{d}{dz}[z\varphi(z)] - \frac{\dfrac{d}{dz}\left\{\dfrac{1}{z}\dfrac{d}{dz}[z\varphi(z)]\right\}}{\dfrac{1}{z}\dfrac{d}{dz}[z\varphi(z)]} = 0. \tag{28}$$

Equation (28) can be immediately integrated and its general solution is

$$\phi(z) = \frac{1}{z}(Az^2 + B)^{-3/2}, \tag{29}$$

where A and B are arbitrary integration constants. Notice now that the Newtonian potential analyzed before, which is a solution of Eq. (22), is also a necessarily solution of Eq. (26) as can be immediately verified. Such a solution can be obtained from Eq. (29) by setting $A=0$. Therefore, without any loss of generality, we suppose $A \neq 0$ and for convenience recast Eq. (29) into the form

$$\phi(z) = \frac{4k}{z}(z^2 + \beta^2)^{-3/2}, \tag{30}$$

where $k \neq 0$ and β are different constants. We write β^2 in order to assure that the function $\phi(z)$, and consequently the corresponding potential function, does not become ill defined in the region around $z=0$. Making use of Eqs. (30) and (27), we can determine the potential function $v(z)$. For $\beta=0$, we obtain

$$v(z) = \frac{k}{2z^2} + \gamma z^2 + \delta, \tag{31}$$

and for $\beta \neq 0$

$$v(z) = -\frac{4kz\sqrt{z^2 + \beta^2}}{\beta^4} + \gamma z^2 + \delta. \tag{32}$$

The constant δ is a simple shift of the zero of the potential and does not influence the law of force. Considering first the potential given by Eq. (31) and evaluating the second derivative of the effective potential at the stationary point, we obtain

$$u''(z_0) = \frac{2k}{z_0^4}. \tag{33}$$

It follows that we will have a minimum only if $k > 0$. Making use of Eq. (13), we have

$$\chi(\zeta) = \sqrt{\frac{2}{k}}\frac{z_0^2\zeta}{\zeta + z_0}. \tag{34}$$

Performing the analytical continuation of Eq. (34) and making use of the Cauchy formula, we obtain from Eq. (12)

$$C_n = \frac{1}{n}\left(\frac{2}{k}\right)^{n/2}\frac{1}{2\pi i}z_0^{2n}\oint_{C_{z_0}}\frac{\zeta^n}{(\zeta^2 - z_0^2)^n}d\zeta. \tag{35}$$

The integral in Eq. (35) can be easily evaluated by making the transformation $\eta = \zeta^2 - z_0^2$ in the neighborhood of z_0. It follows then

$$C_n = \frac{1}{n}\left(\frac{2}{k}\right)^{n/2}\frac{1}{2\pi i}z_0^{2n}\oint_{C_{z_0}}\frac{(\eta+z_0^2)_{n-1/2}}{\eta^n}d\eta. \tag{36}$$

The residue of Eq. (36) is obviously null if n is an odd integer greater than one. Therefore, we can be sure that all potentials given by Eq. (31) generate orbits for which the apsidal angles depend only on the angular momentum. The apsidal angle for this class of potentials follows from Eq. (21) and is given by

$$\Delta\theta_a = \frac{\ell\pi}{\sqrt{2m}}C_1 = \frac{\pi}{2}\sqrt{1-\frac{\gamma z_0^4}{2k}}. \tag{37}$$

Recall that z_0 is linked to ℓ. In order to have an apsidal angle not dependent on the angular momentum, it is necessary that the constant γ be set equal to zero. This will lead us to potentials of the form

$$v(z) = \frac{k}{2z^2}, \tag{38}$$

which corresponds to the isotropic harmonic oscillator. In this case, Eq. (37) yields

$$\Delta\theta_a = \frac{\pi}{2}, \tag{39}$$

and, therefore, the orbit is closed.

V. ADDITIONAL THIRD POTENTIAL

Finally let us consider the potential given by Eq. (32) for which the effective potential reads as

$$u(z) = -\frac{\alpha z\sqrt{z^2+\beta^2}}{\beta^4} + \frac{\alpha(4z_0^3+2\beta^2 z_0)z^2}{4\beta^4 z_0^2\sqrt{z_0^2+\beta^2}}. \tag{40}$$

The second derivative of $u(z)$ at z_0 is easily evaluated and the result is

$$u''(z_0) = \frac{\alpha}{z_0\sqrt{(z_0^2+\beta^2)^3}}. \tag{41}$$

We can see that only for $\alpha > 0$ we will have a minimum, and in what follows we will show that this must indeed be the case here. Making use of Eq. (13), we obtain after some algebraic manipulations

$$\chi(\zeta) = \sqrt{\frac{1}{2k}}\sqrt{\sqrt{z_0^4+\beta^2 z_0^2}} \times \frac{\sqrt{(\beta^2+2z_0^2)\zeta^2 + 2\sqrt{z_0^4+\beta^2 z_0^2}\sqrt{\zeta^4+\beta^2\zeta^2}+\beta^2 z_0^2}}{\zeta+z_0}. \tag{42}$$

Next we consider the analytical continuation of this function. The complex function given by Eq. (42) has algebraic branching points at $z=\pm z_0$ and $z=\pm i\beta$ and is defined on a four-sheeted Riemann surface. Hence, we choose a sheet on this Riemann surface such that the functions $\chi(\zeta)$ do not have branch points at $z=\pm z_0$ and with a branch cut along the imaginary axis from $z=i\beta$ to $z=-i\beta$. Notice that in this case, the numerator of Eq. (42) does not have zeros. Making use of Cauchy formula, we can write

$$\left.\frac{d^{n-1}\chi^n(z)}{dz^{n-1}}\right|_{z=z_0} = \frac{(n-1)!}{2\pi i}\oint_{C_{z_0}}\vartheta^n(\zeta)d\zeta, \tag{43}$$

where we have defined

$$\vartheta(\zeta) = \sqrt{\frac{1}{2k}}\frac{\sqrt{z_0^4+\beta^2 z_0^2}\sqrt{(\beta^2+2z_0^2)\zeta^2+2\sqrt{z_0^4+\beta^2 z_0^2}\sqrt{\zeta^4+\beta^2\zeta^2}+\beta^2 z_0^2}}{\zeta^2-z_0^2} \tag{44}$$

and C_{z_0} is a small closed path encircling the point z_0. The integrand in Eq. (43) has another pole at the point $z=-z_0$. Then, choosing the path indicated in Fig. 2, the following results can be easily obtained:

$$\lim_{R\to\infty}\oint_{C_R}\vartheta^n(\zeta)d\zeta = 0, \tag{45}$$

if $n > 1$; and also

$$\lim_{r_2\to 0}\oint_C\vartheta^n(\zeta)d\zeta = 0, \tag{46}$$

and

$$\lim_{r_4\to 0}\oint_C\vartheta^n(\zeta)d\zeta = 0. \tag{47}$$

Finally, the integral over the finite parts of the real axis and the imaginary axis [see Fig. 2] cancel out. Then we have

$$\int_{\zeta=z_0}\vartheta^n(\zeta)d\zeta + \int_{\zeta=-z_0}\vartheta^n(\zeta)d\zeta = 0, \tag{48}$$

for $n > 1$. We can easily prove by making the substitution $\zeta \to -\zeta$ in the second integral above that

$$[1+(-1)^{n+1}]\int_{\zeta=z_0}\vartheta^n(\zeta)d\zeta = 0, \tag{49}$$

and therefore we conclude that

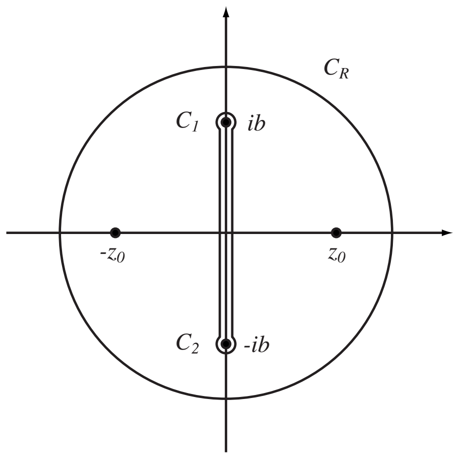

FIG. 2. Contour for the integration of Eq. (45).

$$\left. \frac{d^{n-1}\chi^n(z)}{dz^{n-1}} \right|_{z=z_0} = 0, \tag{50}$$

if n is an odd integer greater than one. The last step is the evaluation of the apsidal angle for this case. Making use of Eqs. (4) and (40), we have

$$\ell^2 = -\frac{mv'(z_0)}{z_0} = -\frac{m}{z_0}\left[-\frac{1}{2}\frac{\alpha(4z_0^3+2\beta z_0)}{\beta^2\sqrt{z_0^4+\beta z_0^2}} - 2\gamma\right]$$
$$= m\left[\frac{\alpha(2z_0^2+\beta)}{\beta^2 z_0\sqrt{z_0^2+\beta}} - 2\gamma\right]. \tag{51}$$

Taking this result into Eq. (21), we obtain

$$\Delta\theta_a = \frac{\pi}{\sqrt{\alpha}}\sqrt{\left[(z_0^2+\beta)\left(\frac{\alpha(2z_0^2+\beta)}{\beta^2} - 2\gamma z_0\sqrt{z_0^2+\beta}\right)\right]}. \tag{52}$$

The potentials given by Eq. (32) generate orbits for which the apsidal angles depend only on the angular momentum. In this third case, it is not possible to find constants β and γ in such a way that those angles do not depend also on the angular momentum.

VI. FINAL REMARKS

In this work, we have introduced a method for the computation of the apsidal angle directly for an arbitrary central field of force and give explicit expressions for certain particular cases. Equation (21) is the result of these calculations. We have derived the conditions under which the apsidal angles remain independent of the energy and angular momentum and lead to closed orbits. We have found that only the Newtonian and the isotropic oscillator potentials present such behavior. Moreover, from Eq. (21) we have also calculated explicitly the value of their associated apsidal angles and have found the well-known results $\pi/2$ and π, respectively. We can see also that these two special fields are strongly determined by the dependence of the centrifugal potential on $1/r^2$. As a consequence, we have reobtained Bertrand's theorem. After more than 100 years since its publication, Bertrand's theorem still attracts our attention [12]. Other proofs of Bertrand's theorem, some of them very interesting, (see, for instance, the phase-space approach in [13] or the Hamiltonian one in [14]) can be found in the literature. We have also obtained a third potential which we can lay aside because it does not meet the conditions that lead to bounded closed orbits. Finally, we would like to remark that the apsidal angle as given by its series representation [Eq. (21)], which converges in some neighborhood of U_0 (circular orbits), shows that if all orbits near to the circular one have the same apsidal angle then there are only two possible potentials that admit this fact: the Newtonian and the isotropic harmonic-oscillator one. This is related to a similar result obtained by Féjoz and Kaczmarek [15] concerning periodic orbits close to circular ones.

The analytical function techniques applied to the problem of finding the only central fields that allow for an entire class of bounded closed orbits with a minimum number of restrictions lead in a concise straightforward way directly to the two allowed fields. Equation (21) seems promising and its applications go beyond the rederivation of Bertrand's results. In fact, it can be taken as the starting point in the discussion of problems related to precession phenomena. Work by the present authors in this direction is in progress and results will be published elsewhere.

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